

AUTOMORPHISMS AND QUOTIENTS OF QUATERNIONIC FAKE QUADRICS

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ABSTRACT. A fake quadric is a smooth surface of general type with the same invariants as the quadric in \mathbb{P}^3 , i.e. $c_1^2 = 8$, $c_2 = 4$ and $q = p_g = 0$. We study here quaternionic fake quadrics i.e. fake quadrics constructed arithmetically by using some quaternion algebras over real number fields. We provide examples of quaternionic fake quadrics X with a non-trivial automorphism group and compute the invariants of the minimal desingularisation of the quotient of X by this group. In that way we obtain minimal surfaces Z of general type with $q = p_g = 0$ and $K^2 = 4, 2$ or 1 which contain the maximal number of disjoint (-2) -curves. We then prove that if a surface of general type has the same invariant as Z and same number of (-2) -curves, then we can construct geometrically a surface of general type with $c_1^2 = 8$, $c_2 = 4$.

Key-Words: Surfaces of general type, Fake Quadrics, Automorphisms, Godeaux surfaces, Campedelli surfaces, Surfaces with $q = p_g = 0$.

AMS subject Classification 14J29, 14G35, 11F06, 14J50.

1. INTRODUCTION

A *fake quadric* is a smooth minimal surface of general type with the same numerical invariants as the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ i.e. with Chern numbers $c_1^2 = 8$, $c_2 = 4$ and vanishing geometric genus $p_g = 0$. Two classes of examples of such surfaces are known and these two classes are both quotients of $\mathbb{H} \times \mathbb{H}$, where \mathbb{H} is the upper-half plane, by a cocompact torsion free lattice $\Gamma \subset \text{Aut}(\mathbb{H} \times \mathbb{H})$. In other words, their universal cover is always $\mathbb{H} \times \mathbb{H}$.

The first class of fake quadrics consists of surfaces $X = \Gamma \backslash \mathbb{H} \times \mathbb{H}$ such that the group Γ is reducible. By reducible we mean that there exists a subgroup of finite index $\Gamma' = \Gamma_1 \times \Gamma_2$ of Γ such that the group Γ_i acts on \mathbb{H} and $C_i = \mathbb{H}/\Gamma_i$ is a smooth algebraic curve. This case is now well understood and the full classification of these fake quadrics, called *surfaces isogenous to a higher product*, has been achieved in [3] by Bauer, Catanese and Grunewald. In practice, this classification and construction is done geometrically by classifying triples (C_1, C_2, G) of two smooth curves C_i of general type and an automorphism group G , such that G acts freely on the surface $C_1 \times C_2$ and the quotient $(C_1 \times C_2)/G$ has the asked invariants.

In this paper we will focus on fake quadrics of the second class, that we call *quaternionic fake quadrics*. These fake quadrics are quotients of $\mathbb{H} \times \mathbb{H}$ by cocompact irreducible lattices Γ in $\text{Aut}(\mathbb{H} \times \mathbb{H})$. The lattice Γ is then arithmetic by a theorem of Margulis and is defined by an indefinite quaternion algebra over a totally real number field.

The first quaternionic fake quadrics have been constructed by Shavel [22] in 1978. We know that these surfaces are rigid and thus that there are only a finite number of them, but at the moment we do not have a complete list of all these surfaces. We have a list of “classes” of fake quadrics defined by quaternion algebras over quadratic fields ([7]).

The situation for quaternionic fake quadrics is very similar to the case of fake projective planes which are surfaces of general type with the same numerical invariants as the projective plane. Fake projective planes are all quotients of the 2-dimensional complex unit ball \mathbb{B}^2 by cocompact arithmetic lattices $\Gamma \subset PU(2, 1)$. This provides an arithmetic construction of these surfaces, but it is generally not easy to handle and construct these surfaces geometrically, e.g. as a quotient or ramified cover of some known surfaces.

In order to remedy at this situation, in [11], [13], [14], Keum studied quotients Z of fake projective planes by groups of automorphisms. In this way, he obtained surfaces of general type with geometric genus $p_g = 0$ and was able to rebuild a fake projective plane by only knowing the properties of the quotient surface Z .

The aim of this paper is to study automorphisms of quaternionic fake quadrics and the quotients of these surfaces by groups of automorphisms. Contrary to the case of fake projective planes, the classification of fake quaternionic quadrics is not fully established yet, thus we do not have a complete list of their automorphism groups. The computations in [7] leads us to the conjecture that the order of the automorphism group is less or equal 24 (see Section 4.6).

The first main result we obtain is the following:

Theorem A. *Let $X = \Gamma \backslash \mathbb{H} \times \mathbb{H}$ be a quaternionic fake quadric. An automorphism of X has only finitely many fixed points. There exist fake quaternionic quadrics X with automorphism group isomorphic to*

$$\mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, (\mathbb{Z}/2\mathbb{Z})^3, \mathbb{D}_6, \mathbb{D}_{10}, \mathbb{D}_4 \times \mathbb{Z}/2\mathbb{Z} \text{ or } \mathbb{D}_6 \times \mathbb{Z}/2\mathbb{Z},$$

where \mathbb{D}_n is the dihedral group with order $2n$.

Note that the knowledge of surfaces of general type with $p_g = 0$ with a large automorphism group can be interesting to check whether the Bloch conjecture holds (see e.g. [10]). We then study the minimal desingularisation of the quotient of a quaternionic fake quadric by a group of automorphisms:

Theorem B. *Let X be a quaternionic fake quadric and G a finite group of automorphisms of X . The minimal desingularisation Z of the quotient X/G has the following numerical invariants:*

G	$c_1^2(Z)$	$c_2(Z)$	Singularities on X/G	Minimal	$\kappa(Z)$
$\mathbb{Z}/2\mathbb{Z}$	4	8	$4A_1$	yes	2
$(\mathbb{Z}/2\mathbb{Z})^2$	2	10	$6A_1$	yes	2
$(\mathbb{Z}/2\mathbb{Z})^3$	1	11	$7A_1$	yes	2
$\mathbb{Z}/3\mathbb{Z}$	2	10	$2A_{3,1} + 2A_2$	-	2
\mathbb{D}_3	1	11	$4A_1 + A_2 + A_{3,1}$	-	2
$\mathbb{Z}/6\mathbb{Z}$	-4	16	$2A_{6,1} + 2A_{6,5}$	no	-
$\mathbb{Z}/10\mathbb{Z}$	-12	24	$2A_{10,1} + 2A_{10,9}$	no	-

Here, κ indicates the Kodaira dimension of the surface Z .

We obtain also results and restrictions for the groups $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$ and \mathbb{D}_5 . We note that the surfaces general type we obtain have vanishing geometric genus. The classification of surfaces with $p_g = 0$ is not established and we intend to compute the fundamental groups of our examples in a forthcoming paper.

A curve C on a surface is called nodal if $C \simeq \mathbb{P}^1$ and $C^2 = -2$. A nodal curve is the resolution of a nodal singularity. The surfaces Z we obtain as quotient of a fake quadric by an automorphism group $(\mathbb{Z}/2\mathbb{Z})^n$, $n \in \{1, 2, 3\}$ have the maximum number of nodal curves (Miyaoaka bound [18]). For the case $c_1^2(Z) = 1$, (i.e. a numerical Godeaux surface) as far as we know, this is the first example of a surface containing the maximum number of (-2) -curves. Interestingly, in [12] Remark 4.10, Keum conjectured its existence as a quotient of a fake quadric. If minimal, the surfaces obtained by quotient by the groups $\mathbb{Z}/3\mathbb{Z}$ and \mathbb{D}_3 have also the maximum number of quotient singularities. As Keum did with fake planes, we can reverse the construction:

Proposition C. *Let Z be a smooth minimal surface of general type with $q = p_g = 0$.*

- a) Suppose that $c_1^2 = 4, 2$ or 1 , $\text{Pic}(Z)$ has no 2-torsion, and that there is a birational map $Z \rightarrow Y$ onto a surface containing $8 - c_1^2$ nodal singularities A_1 . There exist a smooth minimal surface of general type S with invariants $c_1^2 = 2c_2 = 8$ and a $(\mathbb{Z}/2\mathbb{Z})^m$ -cover $S \rightarrow Y$ ramified over the nodes, with m such that $2^m = \frac{8}{c_1^2}$.*
- b) Suppose that $c_1^2 = 2$, $\text{Pic}(Z)$ has no 3-torsion, and that there is a birational map $Z \rightarrow Y$ onto a surface with $2A_{3,1} + 2A_2$ singularities. There exist a smooth surface S with invariants $c_1^2 = 2c_2 = 8$ and a $(\mathbb{Z}/3\mathbb{Z})$ -cover $Z \rightarrow Y$ ramified over the singularities of Y .*

The proof of part a) of this Proposition uses mainly the results of Dolgachev, Mendes Lopes, Pardini ([6]) and illustrates their theory. Note that [6] gives an example of a surface with $c_1^2 = 4, p_g = 0$ and 4 nodal curves and such that the associated double cover is isogenous to a product. The proof of part b) of the above Proposition is more original because it mixes two types of singularities.

The paper is structured as follows: we begin to recall the known facts on quaternionic fake quadrics, and on quotients of surfaces. We then provide examples of fake quadrics having a large group of automorphisms, compute the quotients surfaces and then reverse the construction on the opposite direction : starting with a surface with the same invariants as the quotient, we construct a surface with $c_1^2 = 2c_2 = 8$.

Acknowledgements. Part of this research was done during the second author stay in Strasbourg University. We thank Margarida Mendes Lopes and Rita Pardini for useful discussions.

2. GENERALITIES ON QUATERNIONIC FAKE QUADRICS

Let us give a more detailed description of quaternionic fake quadrics. First, recall that a lattice $\Gamma < \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \cong \text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ is irreducible if it is not commensurable with a product $\Gamma_1 \times \Gamma_2$ of two discrete subgroups $\Gamma_1, \Gamma_2 \subset \text{PSL}_2(\mathbb{R})$. Equivalently, the image of Γ under the projection onto one of the factors $\text{PSL}_2(\mathbb{R})$ is a dense subgroup of $\text{PSL}_2(\mathbb{R})$. By a famous result of Margulis, an irreducible

lattice Γ in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ is an arithmetic group, and can therefore be described in the following way:

There exists a totally real number field k of degree $g = [k : \mathbb{Q}] \geq 2$ and a quaternion algebra $B = (\alpha, \beta)_k := \langle 1, i, j, ij \rangle_k$ with $i^2 = \alpha \in k, j^2 = \beta \in k, ij = -ji$, over k such that

$$(2.1) \quad B \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\rho \in \mathrm{Hom}(k, \mathbb{R})} B^{\rho} \cong M_2(\mathbb{R}) \times M_2(\mathbb{R}) \times \underbrace{H_{\mathbb{R}} \times \dots \times H_{\mathbb{R}}}_{g-2}.$$

Here, $B^{\rho} = (\alpha^{\rho}, \beta^{\rho})_{\mathbb{R}}$ and $H_{\mathbb{R}} = (-1, -1)_{\mathbb{R}}$ denotes the skew field of Hamilton quaternions. Let \mathcal{O}_k be the ring of integers of k and \mathcal{O} a maximal order in B , i.e. a subring of B which is a full \mathcal{O}_k -lattice in B . Finally, let $\mathcal{O}^{(1)}$ be the subgroup of all elements in \mathcal{O} of reduced norm one.

The isomorphism (2.1) induces an embedding of $\mathcal{O}^{(1)}$ into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ by taking the element $\gamma \in \mathcal{O}^{(1)}$ to the pair $(\gamma^{\rho_1}, \gamma^{\rho_2}) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, where γ^{ρ_i} is the image of γ in B^{ρ_i} . The group $\mathcal{O}^{(1)}$ then acts on $\mathbb{H} \times \mathbb{H}$ as a group of fractional linear transformations. Namely, if $(z, w) \in \mathbb{H} \times \mathbb{H}$ is a point and an element $\gamma \in \mathcal{O}^{(1)}$ is identified with two matrices γ^{ρ_1} and $\gamma^{\rho_2} \in \mathrm{SL}_2(\mathbb{R})$, then

$$\gamma(z, w) = (\gamma^{\rho_1} z, \gamma^{\rho_2} w).$$

After dividing out by the ineffective kernel, one considers the group

$$\Gamma_{\mathcal{O}}^1 = \mathcal{O}^{(1)} / \{\pm 1\} \subset \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$$

and it can be proven that $\Gamma_{\mathcal{O}}^1$ is an irreducible lattice in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ (see [16]). In general we say that a subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ is an *arithmetic lattice* if there exists $k, B, \rho_1, \rho_2, \mathcal{O}$ as above such that Γ is commensurable with $\Gamma_{\mathcal{O}}^1$.

Let Γ be irreducible and $X_{\Gamma} := \Gamma \backslash \mathbb{H} \times \mathbb{H}$ be the orbit space of the discontinuous action of Γ on $\mathbb{H} \times \mathbb{H}$. Then, there is a natural structure of compact algebraic surface on X_{Γ} and X_{Γ} is smooth if and only if Γ is torsion free. The numerical invariants of a smooth X_{Γ} are computed in [16], see also [22]. It follows that X_{Γ} is a fake quadric if and only if $c_2(X_{\Gamma}) = 4$ (see [22]).

3. GENERALITIES ON QUOTIENTS OF A SURFACE

In this section we recall results from the theory of singularities and on the resolution of the quotient of a surface by a group. The main reference for these topics is [2], see also [21].

Let us denote by G an automorphism group acting on S , by $X = S/G$ the quotient surface and by $\pi : Z \rightarrow S/G$ the minimal desingularisation map.

Proposition 3.1 (Topological Lefschetz formula). *Let σ be an automorphism acting on S and S^{σ} the fixed point set of σ . We have*

$$e(S^{\sigma}) = \sum_{j=0}^{j=4} (-1)^j \mathrm{Tr}(\sigma | H^i(S, \mathbb{Z})_{mt})$$

where $H^i(S, \mathbb{Z})_{mt}$ is the group $H^i(S, \mathbb{Z})$ modulo torsion.

Note that for a fake quadric S we have $q = p_g = 0$, thus

$$H^1(S, \mathbb{Z})_{mt} = \{0\}, \quad H^2(S, \mathbb{Z}) \otimes \mathbb{C} = H^1(S, \Omega_S).$$

Corollary 3.2. *Let S be a fake quadric and σ an automorphism of order $n > 1$ acting on S . We have $e(S^\sigma) = 2$ or 4 . If n is prime to 2 we have $e(S^\sigma) = 4$.*

Proof. For a fake quadric, the space $H^1(S, \Omega_S)$ is 2-dimensional and is generated by the classes of 2 curves in the Néron-Severi group. As an automorphism preserves the canonical divisor, the invariant subspace of $H^1(S, \Omega_S)$ is at least 1 dimensional. Therefore the trace of σ on $H^1(S, \Omega_S)$ is 2 or 0. If we suppose that this action is not trivial, then 2 divides the order of σ . \square

Let ξ be a primitive n^{th} -root of unity. Let us recall that for $1 \leq q \leq n-1$ coprime to n , the quotient of \mathbb{C}^2 by the action of

$$(x, y) \rightarrow (\xi x, \xi^q y)$$

has a unique singularity, called a $A_{n,q}$ singularity. For $n, m > 0$ two numbers, we denote $[n, m] = n - \frac{1}{m}$. A $A_{n,q}$ singularity is resolved by a chain of smooth rational curves C_1, \dots, C_k such that C_i cuts $C_{i\pm 1}$ for $2 \leq i \leq k-1$ and $C_i^2 = -n_i$ for integers $n_i \geq 2$ determined by the relation:

$$\frac{n}{q} = [n_1, [n_2, \dots, [n_{k-1}, n_k]] \dots].$$

We denote classically $A_{n,n-1} = A_{n-1}$.

Let S be a surface with $p_g = q = 0$ and let σ be an order $n \geq 2$ automorphism such that the fixed points of the σ^k , $k = 1, \dots, n-1$ are isolated.

Proposition 3.3. *(Holomorphic Lefschetz fixed point formula, [1] p. 567). Let S^σ be the fixed point set of σ . Then*

$$1 = \sum_{s \in S^\sigma} \frac{1}{\det(1 - d\sigma|_{T_{S,s}})},$$

where $d\sigma_s|_{T_{S,s}}$ denotes the action of σ on the tangent space $T_{S,s}$.

Suppose moreover that the automorphism σ has prime order p . Let ξ be a primitive p^{th} -root of unity. Let r_i be the number of isolated fixed points of σ whose image in S/σ are $A_{p,i}$ singularities.

Proposition 3.4. *(Zhang's formula, [27] Lemma 1.6). We have:*

$$\sum_{i=1}^{i=p-1} r_i a_i(p) = 1$$

where

$$a_i(p) = \frac{1}{p-1} \sum_{j=1}^{j=p-1} \frac{1}{(1 - \xi^j)(1 - \xi^{ij})}$$

In particular, we have :

$$a_1(p) = \frac{5-p}{12}, \quad a_2(p) = \frac{11-p}{24}, \quad a_3(5) = \frac{1}{4}, \quad a_4(5) = \frac{1}{2}.$$

Let $1 \leq i < p$ and $1 \leq k < p$ be such that $ik = 1 \pmod{p}$. As $A_{p,i} = A_{p,k}$, the notations for r_i and r_k in Zhang's Lemma can be confusing. However, as $a_i(p) = a_k(p)$, there should be no trouble in taking the convention that $r_i + r_k$ is the total number of $A_{p,i} = A_{p,k}$ singularities, rather than choosing a representative i or k for every such pair (i, k) .

Let us recall that an automorphism of a vector space is called a reflection if all its eigenvalues but one are equal to 1. Let S be a surface and G an automorphism group acting on S . Suppose that for every automorphism of G the fixed point set is finite. Let s be a fixed point of G ; recall (see [2]):

Lemma 3.5. *The action of the group G on the tangent space $T_{S,s}$ is faithful and has no reflection.*

In particular, if G is cyclic of order n , the singularity type of the image of the fixed point s in the quotient S/G is always a $A_{n,q}$ with q prime to n .

Lemma 3.6. *The Euler number of S/G is given by the formula*

$$e(S/G) = \frac{1}{|G|}(e(S) + \sum_{n \geq 2} (n-1)e(S_n)),$$

where $S_n = \{s \in S \mid |\text{Stab}(G, s)| = n\}$. The Euler number of the minimal resolution Z is the sum of $e(S/G)$ and the number of irreducible components of the exceptional curves of the resolution $\pi : Z \rightarrow S/G$.

Let C_1, \dots, C_k be the irreducible components of the one dimensional fibers of $\pi : Z \rightarrow X = S/G$. We have the relations $K_Z = \pi^* K_X - \sum_{i=1}^k a_i C_i$, for rational numbers a_i such that $K_Z C_k = -2 - C_k^2$ and $C_k \pi^* K_X = 0$.

Moreover : $K_X^2 = \frac{K_S^2}{|G|}$ where $|G|$ is the order of G . As K_S ample, the canonical \mathbb{Q} -divisor K_X is ample and $\pi^* K_X$ is nef. We remark also that $K_Z^2 \leq K_X^2$.

Lemma 3.7. *Let S be a surface with $q = p_g = 0$. The minimal resolution of the quotient of S by a group G has always $q = p_g = 0$.*

Let us now specialize to surfaces $X_\Gamma = \mathbb{H} \times \mathbb{H} / \Gamma$ where Γ is a cocompact and irreducible torsion-free lattice. Let $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ be the involution exchanging the two factors. The group $\text{Aut}(\mathbb{H} \times \mathbb{H})$ is the semi-direct product of $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ by the group generated by μ . Let $\Gamma < \text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ be a cocompact torsion-free lattice and $X_\Gamma = \Gamma \backslash \mathbb{H} \times \mathbb{H}$, then, since $\mathbb{H} \times \mathbb{H}$ is the universal covering of X_Γ , every automorphism σ of X_Γ lifts to an automorphism $\tilde{\sigma}$ of $\mathbb{H} \times \mathbb{H}$. If $\tilde{\sigma}$ is in $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ it obviously normalizes Γ . Therefore, the factor preserving automorphism group of X_Γ is $N\Gamma/\Gamma$, where $N\Gamma$ is the normalizer of Γ in $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$. Altogether, every automorphism is either represented by a coset $\gamma\Gamma$ with $\gamma \in N\Gamma$ or is of type $(\gamma\Gamma) \circ \mu$. The following result is a key for our computations:

Theorem 3.8. *An automorphism σ of X_Γ has only finitely many fixed points or σ is an involution whose fixed point set is purely one-dimensional. The latter never happens for a quaternionic fake quadric.*

Proof. Suppose that γ is in the subgroup $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$. Then, as explained above, σ can be represented by a coset $\gamma\Gamma$ with $\gamma \in N\Gamma$, where $N\Gamma$ is the normalizer of Γ in G . It is sufficient to show that γ as a mapping $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$

has only finitely many fixed points in $\mathbb{H} \times \mathbb{H}$ modulo the action of Γ . Assume that $\gamma(z, w) = (\gamma^{\rho_1} z, \gamma^{\rho_2} w) = (z, w)$. Then γ is an elliptic transformation, i.e. $4 \det(\gamma) - \text{Tr}(\gamma^{\rho_i})^2 > 0$ for $i = 1, 2$. The reason is the following: if (z, w) is a fixed point, then in particular z is a fixed point of γ^{ρ_1} and w is a fixed point of γ^{ρ_2} . The only automorphisms of \mathbb{H} with fixed points in \mathbb{H} are elliptic transformations. Then, by definition γ is elliptic.

Every non-trivial elliptic transformation of \mathbb{H} has a unique fixed point in \mathbb{H} (the eigenvalue of the matrix which has the positive imaginary part). Moreover, because Γ is irreducible, γ^{ρ_1} is non-trivial if and only if γ^{ρ_2} is non trivial (observe here that if Γ were not irreducible, there can be automorphism of the form $(\gamma_1, 1)$ that has non-isolated fixed point).

Let thus (z, w) be the unique fixed point of γ in $\mathbb{H} \times \mathbb{H}$. The $N\Gamma$ -orbit of (z, w) is discrete in $\mathbb{H} \times \mathbb{H}$, therefore there is only one representative of (z, w) modulo the action of $N\Gamma$. Now, $N\Gamma$ is a finite index extension of Γ , therefore there are only finitely many representatives of (z, w) modulo Γ .

Let us now suppose that the automorphism σ is represented by $\gamma\mu \in \text{Aut}(\mathbb{H} \times \mathbb{H})$. Then $(\gamma\mu)^2 = (\gamma^{\rho_1}\gamma^{\rho_2}, \gamma^{\rho_2}\gamma^{\rho_1})$ is an element of $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ that acts on the surface. Suppose that σ has an infinite number of fixed points, then σ^2 must be the identity and $\gamma^{\rho_1}\gamma^{\rho_2}$ must be in Γ . A fixed point (z, w) satisfy:

$$(\gamma^{\rho_1} w, \gamma^{\rho_2} z) = \lambda(z, w)$$

for a $\lambda \in \Gamma$. Up to the change of γ by $\lambda^{-1}\gamma$, we can suppose that $\lambda = 1$, thus $w = \gamma^{\rho_2} z$ and $\gamma^{\rho_1}\gamma^{\rho_2} = 1$ because $\gamma^{\rho_1}\gamma^{\rho_2}$ is in Γ that is torsion free. Reciprocally, let be $t \in \mathbb{H}$; as $\gamma^{\rho_1}\gamma^{\rho_2} = 1$ the point $(t, \gamma^{\rho_2} t)$ satisfy

$$\gamma\mu(t, \gamma^{\rho_2} t) = (t, \gamma^{\rho_2} t).$$

Therefore there is no isolated fixed points for σ .

Assume now that X_Γ is a quaternionic fake quadric. The fixed locus C of σ is a smooth curve. The topological Lefschetz formula (see Corollary 3.2) implies that the genus of the irreducible components of C is negative, thus the automorphism has only a finite number of fixed points. \square

4. QUATERNIONIC FAKE QUADRICS WITH NON-TRIVIAL AUTOMORPHISM GROUPS.

As already mentioned, a series of examples of quaternionic fake quadrics has been constructed by I. Shavel in [22] (see also [23]). There, the author concentrates on arithmetic lattices $\Gamma \supseteq \Gamma_{\mathcal{O}}^1$ which are defined by quaternion algebras over real quadratic fields of class number one. More recently, in [7], more examples of quaternionic quadrics associated with quaternion algebras over quadratic fields have been found. In this section we will list all known examples of quaternionic fake quadrics together with their automorphism groups. We refer the reader to [26] for generalities on quaternion algebras.

Let us first make a few general observations, before we discuss the examples in detail. For technical reasons it is more practical to consider the group $\text{PGL}_2^+(\mathbb{R}) \times \text{PGL}_2^+(\mathbb{R})$, where $\text{PGL}_2^+(\mathbb{R}) = \text{GL}_2^+(\mathbb{R})/\mathbb{R}^*$ and $\text{GL}_2^+(\mathbb{R})$ is the group of all 2×2 matrices with positive determinant, instead of $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$. We identify

$\mathrm{PGL}_2^+(\mathbb{R}) \times \mathrm{PGL}_2^+(\mathbb{R})$ with the group $\mathrm{Aut}\mathbb{H} \times \mathrm{Aut}\mathbb{H}$ of holomorphic automorphisms which preserve the two factors.

In point of view of Theorem 3.8 it is more interesting to consider the automorphism subgroups $G \leq N\Gamma/\Gamma =: \mathrm{Aut}(X_\Gamma)$ of factor preserving automorphisms, which we will do in the following. Since fake quadrics X_Γ have relatively small co-volume, they tend to be large groups and therefore the order $|\mathrm{Aut}(X_\Gamma)| = |N\Gamma/\Gamma|$ is not too big. The normalizers $N\Gamma$ will be maximal lattices and all such lattices can be described arithmetically as follows (see [5]).

If X_Γ is a quaternionic fake quadric, there is an associated tuple $(k, \rho_1, \rho_2, B, \mathcal{O})$ as described in Section 2. The quaternion algebra B is for fixed ρ_1, ρ_2 uniquely determined (up to isomorphism) by the reduced discriminant $d_B = v_1 \dots v_r$, the formal product over finite places v_i of k where B is ramified, i.e. $B \otimes_k k_{v_i} \not\cong M_2(k_{v_i})$, hence $(k, \rho_1, \rho_2, B, \mathcal{O}) = (k, \rho_1, \rho_2, d_B, \mathcal{O})$. In the following we will often abbreviate such a datum which determines the quaternion algebra B with $B(k, d_B)$ or $B(k, v_1 \dots v_r)$. Let us fix such a datum $B(k, v_1 \dots v_r)$ and let B^+ be the group of all $x \in B^*$ such that the reduced norm $\mathrm{Nrd}(x)$ is totally positive. It is known that

$$(4.1) \quad N\Gamma_{\mathcal{O}}^+ = \{x \in B^+ \mid x\mathcal{O}x^{-1} = \mathcal{O}\}/k^*$$

is a maximal lattice. Obviously $N\Gamma_{\mathcal{O}}^+$ contains $\Gamma_{\mathcal{O}}^1$ and it is known that $\Gamma_{\mathcal{O}}^1$ is normal in $N\Gamma_{\mathcal{O}}^+$ with $N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1 \cong (\mathbb{Z}/2\mathbb{Z})^l$ an elementary abelian 2-group with $l \geq r$ and r is the number of ramified places in B (see [7] and references therein). It follows that a quaternionic fake quadric X_Γ with $\Gamma \supseteq \Gamma_{\mathcal{O}}^1$ will have an elementary abelian 2-group as the automorphism group $\mathrm{Aut}(X_\Gamma)$. All Shavel's examples will provide such automorphism groups.

4.1. A fake quadric with automorphism group $\mathbb{Z}/2\mathbb{Z}$. There are examples of quaternionic fake quadrics X_Γ whose automorphism group is $\mathbb{Z}/2\mathbb{Z}$ and, as mentioned, they already appear in [22].

For example, let $k = \mathbb{Q}(\sqrt{2})$ and let $B = B(k, \mathfrak{p}_3\mathfrak{p}_7)$ be the (unique) quaternion algebra over k which is ramified exactly at the two finite primes \mathfrak{p}_3 and \mathfrak{p}_7 of k lying over the rational primes 3 and 7 respectively. Since k has the class number one, there is the unique (up to conjugation) maximal order \mathcal{O} in B . Consider the group $\Gamma_{\mathcal{O}}^1$. By [22], Proposition 4.7, $X_{\Gamma_{\mathcal{O}}^1}$ is smooth. By the already mentioned general result of Matsushima and Shimura [16], $q(X_{\Gamma_{\mathcal{O}}^1}) = 0$. The Euler number $c_2(X_{\Gamma_{\mathcal{O}}^1})$ is computed via the volume formula of Shimizu (see [22], Theorem 3.1). Since the prime 3 is inert and 7 is decomposed in k , this formula gives $c_2(X_{\Gamma_{\mathcal{O}}^1}) = 8$. The normalizer of $\Gamma_{\mathcal{O}}^1$ is $N\Gamma_{\mathcal{O}}^+$ and by [22], Proposition 1.3 and 1.4 we have

$$\mathrm{Aut}(X_{\Gamma_{\mathcal{O}}^1}) \cong L_1/L_2 = \langle [\mathfrak{p}_3], [\mathfrak{p}_7] \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

where L_1 is the group of principal fractional ideals of type $(\mathfrak{p}_3)(\mathfrak{p}_7)I^2$ (I a principal ideal in k) for which one can find a totally positive generator and L_2 consists of all principal ideals of type (a^2) with $a \in k$ (See also [24], 3.12). Let $\Gamma_{\mathfrak{p}_3}$ be the kernel of the canonical homomorphism

$$N\Gamma_{\mathcal{O}}^+ \longrightarrow L_1/L_2 \longrightarrow \langle [\mathfrak{p}_7] \rangle.$$

By Shavel's criterion (see [22], Theorem 4.11) $\Gamma_{\mathfrak{p}_3}$ is torsion free and as $[\Gamma_{\mathfrak{p}_3} : \Gamma_{\mathcal{O}}^1] = 2$, $X_{\Gamma_{\mathfrak{p}_3}}$ is a fake quadric with $\mathrm{Aut}(X_{\Gamma_{\mathfrak{p}_3}}) \cong \mathbb{Z}/2\mathbb{Z}$.

4.2. A fake quadric with automorphism group $(\mathbb{Z}/2\mathbb{Z})^2$. Consider again $k = \mathbb{Q}(\sqrt{2})$ and now the quaternion algebra $B = B(k, \mathfrak{p}_2 \mathfrak{p}_5)$ over k which is ramified exactly at the two finite places \mathfrak{p}_2 and \mathfrak{p}_5 . Again there is the unique maximal order \mathcal{O} in B and as in the previous example, Shavel's results show that $X_{\Gamma_{\mathcal{O}}^1}$ is smooth. The prime 2 is ramified and 5 is inert in k and therefore Shimizu's volume formula gives $c_2(X_{\Gamma_{\mathcal{O}}^1}) = 4$. Hence $X_{\Gamma_{\mathcal{O}}^1}$ is a fake quadric. With the same arguments as in the previous example $\text{Aut}(X_{\Gamma_{\mathcal{O}}^1})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

4.3. A fake quadric with automorphism group \mathbb{D}_{10} . Consider $k = \mathbb{Q}(\sqrt{5})$ and the quaternion algebra $B = B(k, \mathfrak{p}_2 \mathfrak{p}_5)$ over k which is ramified exactly at the primes \mathfrak{p}_2 and \mathfrak{p}_5 . In this case the group $\Gamma_{\mathcal{O}}^1$, where \mathcal{O} is again a maximal order in B , contains torsion elements of order 5 and no other torsions (see [22], Proposition 4.7 and Theorem 4.8)¹. Volume formula of Shimizu gives in this case $c_2(X_{\Gamma_{\mathcal{O}}^1}) = 4/5$. Let us now give a torsion-free subgroup $\Gamma < \Gamma_{\mathcal{O}}^1$ of index 5. The corresponding surface X_{Γ} will be a fake quadric. Since \mathfrak{p}_2 is ramified in B , there is a prime ideal Π_2 in \mathcal{O} lying over \mathfrak{p}_2 and satisfying $\Pi_2^2 = \mathfrak{p}_2$. Let

$$\Gamma = \Gamma_{\mathcal{O}}^1(\Pi_2) = \{x \in \Gamma_{\mathcal{O}}^1 \mid x \equiv 1 \pmod{\Pi_2}\}.$$

$\Gamma_{\mathcal{O}}^1(\Pi_2)$ is a normal subgroup in $\Gamma_{\mathcal{O}}^1$ and the index can be computed via the localization of B at \mathfrak{p}_2 . Namely, observe first that $\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_2)$ is isomorphic to the factor group $\mathcal{O}^1/\mathcal{O}^1(\Pi_2)$, where

$$\mathcal{O}^1(\Pi_2) = \{x \in \mathcal{O}^1 \mid x \equiv 1 \pmod{\Pi_2}\}.$$

This is because -1 is in $\mathcal{O}^1(\Pi_2)$. By the strong approximation property, the latter factor group is isomorphic to $B_{\mathfrak{p}_2}^1/B_{\mathfrak{p}_2}^1(\Pi_2)$ where $B_{\mathfrak{p}_2}^1$ is the norm-1 group of the local algebra $B_{\mathfrak{p}_2} = B \otimes_k k_{\mathfrak{p}_2}$ and $B_{\mathfrak{p}_2}^1(\Pi_2)$ is its first congruence subgroup. Let $\mathcal{O}_{\mathfrak{p}_2}$ be the maximal order in $B_{\mathfrak{p}_2}$, i.e. $\mathcal{O}_{\mathfrak{p}_2} = \mathcal{O} \otimes_{\mathcal{O}_k} \mathcal{O}_{k_{\mathfrak{p}_2}}$, where $\mathcal{O}_{k_{\mathfrak{p}_2}}$ is the ring of integers in $k_{\mathfrak{p}_2}$. Its maximal ideal is the topological closure of Π_2 and by abuse of notation we will write Π_2 to denote also this maximal ideal. Note that $B_{\mathfrak{p}_2}^1 = \mathcal{O}_{\mathfrak{p}_2}^1$. We use a theorem of C. Riehm (see [20], Theorem 7) by which

$$B_{\mathfrak{p}_2}^1/B_{\mathfrak{p}_2}^1(\Pi_2) \cong \ker((\mathcal{O}_{\mathfrak{p}_2}/\Pi_2)^* \xrightarrow{Nr} (\mathcal{O}_{k_{\mathfrak{p}_2}}/\mathfrak{p}_2)^*) \cong \ker(\mathbb{F}_{16}^* \xrightarrow{Nr} \mathbb{F}_4^*) \cong \mathbb{Z}/5\mathbb{Z}$$

(Note here that the norm map induces a surjective homomorphism of multiplicative groups). Since $\Gamma_{\mathcal{O}}^1(\Pi_2)$ is embedded in $B_{\mathfrak{p}_2}(\Pi_2)/\pm 1$ and the latter group is a pro-2-group (again by [20]) it can not contain elements of order 5. Therefore, $\Gamma_{\mathcal{O}}^1(\Pi_2)$ is a torsion-free group and $X_{\Gamma_{\mathcal{O}}^1(\Pi_2)}$ is a fake quadric. Obviously, $X_{\Gamma_{\mathcal{O}}^1(\Pi_2)}$ contains an automorphism of order 5 coming from the 5-torsion in $\Gamma_{\mathcal{O}}^1$. In order to determine the full automorphism group $\text{Aut}(X_{\Gamma_{\mathcal{O}}^1(\Pi_2)})$ we first need to find the normalizer of $\Gamma_{\mathcal{O}}^1(\Pi_2)$. It is not difficult to see that $\Gamma_{\mathcal{O}}^1(\Pi_2)$ is a normal subgroup in $N\Gamma_{\mathcal{O}}^+$ which is maximal, hence the normalizer $\text{Aut}(X_{\Gamma_{\mathcal{O}}^1(\Pi_2)})$ is a part of the exact sequence

$$(4.2) \quad 1 \longrightarrow \Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_2) \longrightarrow N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1(\Pi_2) \longrightarrow N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1 \longrightarrow 1$$

¹ Note that the symbol $\left(\frac{d}{p}\right)$ in Theorem 4.8 of [22] for $p = 2$ should be read as the Kronecker symbol, i.e. $\left(\frac{d}{2}\right) = 1 \Leftrightarrow d \equiv \pm 1 \pmod{8}$ and $= -1 \Leftrightarrow d \equiv \pm 3 \pmod{8}$.

Here, $\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_2) \cong \mathbb{Z}/5\mathbb{Z}$ is generated by the class of a primitive fifth root of unity ξ_5 . This can be seen locally:

We have the isomorphism

$$\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_2) \cong B_{\mathfrak{p}_2}^1/B_{\mathfrak{p}_2}^1(\Pi_2) \cong \ker((\mathcal{O}_{\mathfrak{p}_2}/\Pi_2)^* \longrightarrow (\mathcal{O}_{k_{\mathfrak{p}_2}}/\mathfrak{p}_2)^*) \cong \ker(\mathbb{F}_{16}^* \longrightarrow \mathbb{F}_4^*).$$

From the general theory we have the identification $\mathcal{O}_{\mathfrak{p}_2}/\Pi_2 = \mathcal{O}_{L_{\mathfrak{p}_2}}/\mathfrak{p}_2$, where $L_{\mathfrak{p}_2}$ denotes the unique unramified quadratic extension of $k_{\mathfrak{p}_2} = \mathbb{Q}_2(\sqrt{5})$. By the general theory of cyclotomic fields, $k_{\mathfrak{p}_2}(\xi_5) = L_{\mathfrak{p}_2}$ and since $\text{Gal}(L_{\mathfrak{p}_2}/k_{\mathfrak{p}_2}) = \langle \xi_5 \mapsto \xi_5^{-1} \rangle$, ξ_5 generates the kernel

$$\ker((\mathcal{O}_{L_{\mathfrak{p}_2}}/\mathfrak{p}_2)^* \longrightarrow (\mathcal{O}_{k_{\mathfrak{p}_2}}/\mathfrak{p}_2)^*).$$

On the other hand $N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ can be represented by the classes modulo k^* of elements $\alpha = \sqrt{-\frac{5+\sqrt{5}}{2}}$ and $\beta = \sqrt{-(5+\sqrt{5})}$. This follows from the general description of $N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1$ (see [22]). In order to determine the automorphism group, by the above exact sequence, we need to determine the action of $N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1$ by conjugation on ξ_5 . We do this locally: First we know that $L_{\mathfrak{p}_2} = k_{\mathfrak{p}_2}(\xi_5)$ is the unramified quadratic extension of $k_{\mathfrak{p}_2}$, hence we can write

$$B_{\mathfrak{p}_2} = L_{\mathfrak{p}_2} \oplus \Pi_2 L_{\mathfrak{p}_2},$$

i.e. $L_{\mathfrak{p}_2}$ is the splitting field of $B_{\mathfrak{p}_2}$. From this representation it is clear that an element $A \in B_{\mathfrak{p}_2}^*$ will commute with ξ_5 if and only if $A \in L_{\mathfrak{p}_2}$. Looking at $A = \alpha$ as above, we see that $K_{\mathfrak{p}_2}(\alpha)/k_{\mathfrak{p}_2}$ is unramified, hence $L_{\mathfrak{p}_2} = k_{\mathfrak{p}_2}(\alpha)$ and α commutes with ξ_5 . On the other hand $k_{\mathfrak{p}_2}(\beta)/k_{\mathfrak{p}_2}$ is ramified and therefore $\beta\xi_5 \neq \xi_5\beta$. Altogether, we get

$$\begin{aligned} \text{Aut}^+(X_{\Gamma_{\mathcal{O}}^1(\Pi_2)}) &= \langle \xi_5, \alpha, \beta \mid \xi_5^5 = \alpha^2 = \beta^2 = 1, \alpha\beta = \beta\alpha, \alpha\xi_5 = \xi_5\alpha, \beta\xi_5\beta^{-1} = \xi_5^{-1} \rangle \\ &= \langle x, y \mid x^{10} = y^2 = 1, yxy = x^{-1} \rangle \end{aligned}$$

Here, we set $x = \xi_5\alpha$ and $y = \beta$. Note that the relation $\beta\xi_5\beta = \xi_5^{-1}$ follows automatically from the fact that β induces an automorphism of order two on $\mathbb{Z}/5\mathbb{Z}$. Thus we proved:

Proposition 4.1. *With above notations we have*

$$\text{Aut}(X_{\Gamma_{\mathcal{O}}^1(\Pi_2)}) \cong \mathbb{D}_{10}.$$

4.4. A fake quadric with automorphism group $(\mathbb{Z}/2\mathbb{Z})^3$. We consider $k = \mathbb{Q}(\sqrt{5})$ and $B = B(k, \mathfrak{p}_2\mathfrak{p}_{11})$, the unique quaternion algebra ramified exactly at the primes \mathfrak{p}_2 and \mathfrak{p}_{11} . Since 2 is inert and 11 is decomposed in k , Shimizu's volume formula gives $c_2(X_{\Gamma_{\mathcal{O}}^1}) = \frac{4}{5 \cdot 12}(4-1)(11-1) = 2$ as the value of the second Chern number of the quotient $X_{\Gamma_{\mathcal{O}}^1}$, where again $\Gamma_{\mathcal{O}}^1$ is the norm-1 group of a maximal order in B . As before, results of [22] show that $\Gamma_{\mathcal{O}}^1$ contains only torsion elements of order 2 and no other torsions (Here, observe that 2 is split in $\mathbb{Q}(\sqrt{-15})$, hence there is no element of order 3 in $\Gamma_{\mathcal{O}}^1$, and note that there is no element of order 5 because $11 \equiv 1 \pmod{5}$ which implies that \mathfrak{p}_{11} is split in $k(\xi_5)$). Since \mathfrak{p}_{11} is ramified

in B , there is the unique prime ideal Π_{11} in \mathcal{O} such that $\Pi_{11}^2 = \mathfrak{p}_{11}$. Consider the principal congruence subgroup

$$\Gamma_{\mathcal{O}}^1(\Pi_{11}) = \{x \in \Gamma_{\mathcal{O}}^1 \mid x \equiv 1 \pmod{\Pi_{11}}\}.$$

It is a normal subgroup in $\Gamma_{\mathcal{O}}^1$. The quotient $\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_{11})$ is isomorphic to $\mathcal{O}^1/\pm\mathcal{O}^1(\Pi_{11})$ because $-1 \notin \mathcal{O}^1(\Pi_{11})$. In order to compute the latter quotient we change over to the localization at the prime \mathfrak{p}_{11} . Let

$$B_{\mathfrak{p}_{11}} = B \otimes_k k_{\mathfrak{p}_{11}} = B \otimes_k \mathbb{Q}_{11}.$$

It is the unique division quaternion algebra over \mathbb{Q}_{11} . Let us write $\mathcal{O}_{\mathfrak{p}_{11}} = \mathcal{O} \otimes_{\mathcal{O}_k} \mathbb{Z}_{11}$ for its maximal order. As in the previous example we will abuse the notation and write Π_{11} to denote the prime ideal of $\mathcal{O}_{\mathfrak{p}_{11}}$. We have

$$\mathcal{O}^1/\mathcal{O}^1(\Pi_{11}) \cong B_{\mathfrak{p}_{11}}^1/B_{\mathfrak{p}_{11}}^1(\Pi_{11}).$$

Note that $B_{\mathfrak{p}_{11}}^1 = \mathcal{O}_{\mathfrak{p}_{11}}^1$. By C. Riehm's result, [20], Theorem 7,

$$B_{\mathfrak{p}_{11}}^1/B_{\mathfrak{p}_{11}}^1(\Pi_{11}) \cong \ker((\mathcal{O}_{\mathfrak{p}_{11}}/\Pi_{11})^* \xrightarrow{Nr} (\mathcal{O}_{k_{\mathfrak{p}_{11}}}/\mathfrak{p}_{11})^*) \cong \ker(\mathbb{F}_{121}^* \longrightarrow \mathbb{F}_{11}^*).$$

Since $\mathbb{F}_{121} = \mathbb{F}_{11}(\xi_{12})$, where ξ_{12} denotes a primitive twelfth root of unity we conclude that $B_{\mathfrak{p}_{11}}^1/B_{\mathfrak{p}_{11}}^1(\Pi_{11})$ is isomorphic to $\mu_{12} = \langle \xi_{12} \rangle$. Hence

$$\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_{11}) \cong B_{\mathfrak{p}_{11}}^1/\pm B_{\mathfrak{p}_{11}}^1(\Pi_{11}) \cong \mu_6 = \langle \xi_6 \rangle.$$

Let us now define an intermediate group

$$\Gamma = \{x \in \Gamma_{\mathcal{O}}^1 \mid x \bmod \Pi_{11} \in \langle \xi_6^2 \rangle \subset \mu_6\}.$$

$\Gamma < \Gamma_{\mathcal{O}}^1$ is a subgroup of index 2, hence $c_2(X_{\Gamma}) = 4$. Moreover, Γ is torsion-free since it can not contain elements of order 2. For if an order-two element x is in Γ , then its image $x \bmod \Pi_{11}$ in $\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_{11})$ lies in a cyclic group $\langle \xi_6^2 \rangle$ of order three, hence it must be the identity. But this means that x is in $\Gamma_{\mathcal{O}}^1(\Pi_{11})$. On the other hand $\Gamma_{\mathcal{O}}^1(\Pi_{11})$ is torsion-free because it embeds in a pro-11 group $B_{\mathfrak{p}_{11}}^1(\Pi_{11})/\pm 1$. This contradicts the assumption on x . All this shows that X_{Γ} is a fake quadric.

Proposition 4.2. *Let $N\Gamma_{\mathcal{O}}^+$ be defined as in (4.1). Then $N\Gamma_{\mathcal{O}}^+$ is the normaliser of Γ and $\text{Aut}(X_{\Gamma}) \cong N\Gamma_{\mathcal{O}}^+/\Gamma$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.*

Proof. From the definition of Γ it is clear that Γ is normal in $\Gamma_{\mathcal{O}}^1$. On the other hand, for the same reason as in the previous example, $\Gamma_{\mathcal{O}}^1(\Pi_{11})$ as well as $\Gamma_{\mathcal{O}}^1$ is normal subgroup in $N\Gamma_{\mathcal{O}}^+$. This already implies that Γ is normal in $N\Gamma_{\mathcal{O}}^+$ because any conjugate of Γ will be a subgroup between $\Gamma_{\mathcal{O}}^1(\Pi_{11})$ and $\Gamma_{\mathcal{O}}^1$ of index 2 in $\Gamma_{\mathcal{O}}^1$. There is only one such group, namely Γ , since $\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_{11}) \cong \mathbb{Z}/6\mathbb{Z}$. Similar exact sequence as (4.2) now shows that $N\Gamma_{\mathcal{O}}^+/\Gamma$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ but this product is a direct product ($\mathbb{Z}/2\mathbb{Z}$ has no non-trivial automorphisms), hence $\text{Aut}(X_{\Gamma}) \cong (\mathbb{Z}/2\mathbb{Z})^3$. \square

4.5. A fake quadric with an automorphism group \mathbb{D}_6 . This time we consider the quadratic field $k = \mathbb{Q}(\sqrt{2})$ and the quaternion algebra $B = B(k, \mathfrak{p}_2\mathfrak{p}_3)$. The norm-1 group $\Gamma_{\mathcal{O}}^1$ of a maximal order in B contains torsion elements of order 3, but no elements of order 2, because \mathfrak{p}_3 is decomposed in $k(\sqrt{-1})$. The second Chern number of the quotient $X_{\Gamma_{\mathcal{O}}^1}$ is $c_2(X_{\Gamma_{\mathcal{O}}^1}) = 1/6(9-1) = 4/3$. Let $\Gamma_{\mathcal{O}}^1(\Pi_2)$ be the principal congruence subgroup corresponding to the prime ideal $\Pi_2 \subset \mathcal{O}$,

defined by the relation $\Pi_2^2 = \mathfrak{p}_2 \mathcal{O}$. Again by Riehm's theorem and with arguments as in Section 4.3, $\Gamma_{\mathcal{O}}^1(\Pi_2)$ is torsion free normal subgroup in $\Gamma_{\mathcal{O}}^1$ of index 3, hence $X_{\Gamma_{\mathcal{O}}^1(\Pi_2)}$ is a fake quadric. Automorphism group $\text{Aut}(X_{\Gamma_{\mathcal{O}}^1(\Pi_2)})$ is isomorphic to the factor group

$$N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1(\Pi_2) \cong \mu_3 \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

We need to describe the action of $N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1$ on $\mu_3 = \langle \xi_3 \rangle$. For this we choose the representatives $\alpha = \sqrt{-(2 + \sqrt{2})}$ and $\beta = \sqrt{-3}$ of $N\Gamma_{\mathcal{O}}^+/\Gamma_{\mathcal{O}}^1$. Since α lies in $k(\xi_3)$, α commutes with ξ_3 . On the other hand, \mathfrak{p}_2 is ramified in $k(\beta)$ and therefore $k_{\mathfrak{p}_2}(\beta)/k_{\mathfrak{p}_2}$ is a ramified extension whereas $k_{\mathfrak{p}_2}(\xi_3)$ is unramified extension of $k_{\mathfrak{p}_2}$. For this reason, β does not commute with ξ_3 and therefore $\beta(\text{mod } \Gamma_{\mathcal{O}}^1)$ acts on $\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_2)$ as $\beta\xi_3\beta = \xi_3^{-1}$, the only non-trivial automorphism of $\mathbb{Z}/3\mathbb{Z}$. This gives a presentation

$$\text{Aut}(X_{\Gamma_{\mathcal{O}}^1(\Pi_2)}) = \langle \xi, \alpha, \beta \mid \xi^3 = \alpha^2 = \beta^2 = 1, \alpha\beta = \beta\alpha, \alpha\xi = \xi\alpha, \beta\xi = \xi^2\beta \rangle$$

which may be seen as a presentation of the dihedral group \mathbb{D}_6 of order 12.

Proposition 4.3. *We have: $\text{Aut}(X_{\Gamma_{\mathcal{O}}^1(\Pi_2)}) \cong \mathbb{D}_6$.*

4.6. $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ automorphism groups. There are more examples of quaternionic fake quadrics with a non-trivial automorphism group. For instance, all examples in Shavel's paper have $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ as the full group of automorphisms. As in previous examples one can show that the congruence subgroup $\Gamma_{\mathcal{O}}^1(\Pi_7)$ of a maximal order \mathcal{O} in the quaternion algebra $B(\mathbb{Q}(\sqrt{2}), \mathfrak{p}_2, \mathfrak{p}_7)$ [choosing the appropriate prime lying over 7] will produce a fake quadric with an automorphism of order 4 (note that here $\Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_7) \cong \mathbb{Z}/4\mathbb{Z}$). The full automorphism group in this case is $\text{Aut}(X_{\Gamma_{\mathcal{O}}^1(\Pi_7)}) \cong \mathbb{D}_4 \times \mathbb{Z}/2\mathbb{Z}$. Similarly, the principal congruence subgroup $\Gamma_{\mathcal{O}}^1(\Pi_2\Pi_3)$ of a maximal order \mathcal{O} in $B(\mathbb{Q}(\sqrt{3}), \mathfrak{p}_2, \mathfrak{p}_3)$ will produce a fake quadric $X_{\Gamma_{\mathcal{O}}^1(\Pi_2\Pi_3)}$ whose automorphism group contains $\mathbb{Z}/6\mathbb{Z}$ ($\cong \Gamma_{\mathcal{O}}^1/\Gamma_{\mathcal{O}}^1(\Pi_2\Pi_3)$). The full automorphism group in this case is isomorphic to the product $\mathbb{D}_6 \times \mathbb{Z}/2\mathbb{Z}$ of order 24, the largest known automorphism group of a fake quadric. In fact, to best of our knowledge, the listed automorphism groups and their subgroups are the all known finite groups which can appear as a group of automorphisms of a fake quadric.

5. COMPUTATIONS OF THE QUOTIENT SURFACES

Let S be a quaternionic fake quadric, G a group of automorphisms of S , $X = S/G$ the quotient surface and $\pi : Z \rightarrow S/G$ the minimal desingularisation map.

Lemma 5.1. *If $K_Z^2 = 0$, the surface Z has Kodaira dimension $\kappa \geq 1$. If $K_Z^2 > 0$, the surface Z has Kodaira dimension $\kappa = 2$.*

Proof. (We follow the ideas from [13]). The quotient surface has $q = p_g = 0$ and thus $\chi(\mathcal{O}_Z) = 1$. Let $m \geq 1$ be an integer, then $-mK_Z\pi^*K_{S/G} = -mK_{S/G}^2 = -\frac{8}{|G|}m < 0$, therefore $H^0(Z, -mK_Z) = \{0\}$ for every $m \geq 1$. Let be $m \geq 2$, then by using Serre duality and Riemann-Roch:

$$H^0(Z, mK_Z) = \chi(\mathcal{O}_Z) + \frac{m(m-1)}{2}K_Z^2 + h^1(Z, mK_Z).$$

If $K_Z^2 > 0$, then immediately, Z has general type. If $K_Z^2 = 0$, the surface has $h^0(Z, 2K_Z) \neq 0$ and cannot be rational by Castelnuovo criterion. Moreover, as $\chi = 1$ it cannot be a ruled surface. Suppose that Z is an Enriques surface. As $K_Z^2 = 0$, it is a minimal surface, but this is impossible because $h^0(Z, 3K_Z) \neq 0$; therefore $\kappa > 0$. \square

Let us first study the case where G is generated by an involution σ .

Proposition 5.2. *An involution σ has 4 fixed points. The invariants of Z are :*

$$K_Z^2 = 4, c_2 = 8, q = p_g = 0, h^{1,1} = 5.$$

The surface Z is minimal of general type.

Proof. By Lefschetz formula (Proposition 3.3), $1 = \sum_{s=\sigma(s)} \frac{1}{4}$, therefore σ has 4 fixed points. Their images in S/σ are 4 A_1 singularities, resolved by 4 (-2) -curves on Z . The invariants of Z are easy to compute.

The surface Z is of general type and is minimal because K_Z is the pullback of the nef divisor K_X . \square

Proposition 5.3. *Let be $G = (\mathbb{Z}/2\mathbb{Z})^2$. The quotient surface $X = S/G$ contains 6 A_1 singularities. The surface Z is minimal of general type and has the invariants:*

$$K_Z^2 = 2, c_2 = 10, q = p_g = 0.$$

Proof. A faithful representation of G on a 2-dimensional space contains reflections, therefore by Lemma 3.5, there are no points fixed by the whole G . The group G contains 3 involutions. Each of these involutions has 4 isolated fixed points whose image in X are $2A_1$ singularities. Thus there are $6A_1$ singularities on $X = S/G$ and we have

$$e(Z) = e(S/G) + 6 = \frac{1}{4}(4 + 12) + 6 = 10.$$

Moreover, $K_Z = \pi^* K_{S/G}$ is nef and $K_{S/G}^2 = K_S^2/4 = 2$. By Lemma 3.7, we have $q = p_g = 0$. \square

Proposition 5.4. *Let be $G = (\mathbb{Z}/2\mathbb{Z})^3$. The quotient surface $X = S/G$ contains 7 A_1 singularities. The minimal resolution Z of X is minimal of general type with*

$$K_Z^2 = 1, c_2 = 11, q = p_g = 0.$$

Proof. The 4 fixed points of an involution form an orbit under the action of G . The group G contains 7 involutions, and correspondingly there are 7 A_1 singularities on $X = S/G$. We have

$$e(Z) = 7 + e(S/G) = 7 + \frac{1}{8}(4 + 7 \cdot 4) = 11.$$

Moreover, $K_Z = \pi^* K_{S/G}$ is nef and $K_Z^2 = \frac{8}{8} = 1$. \square

Remark 5.5. The automorphism group of a fake quadric cannot contain $G = (\mathbb{Z}/2\mathbb{Z})^4$ because the involutive automorphisms of G create only A_1 singularities and therefore the canonical divisor of the minimal resolution Z of $X = S/G$ would satisfy $K_Z^2 = K_{S/G}^2 = \frac{K_S^2}{2^4} = \frac{1}{2}$.

Proposition 5.6. *Let $G = \mathbb{Z}/4\mathbb{Z}$. The singularities of the quotient X are $2A_{4,1} + 2A_{4,3}$ or $A_1 + 2A_{4,3}$. The invariants of the resolution Z are*

$$K_Z^2 = 0, c_2 = 12, q = p_g = 0$$

in the first case, and in the second case Z is minimal and satisfies

$$K_Z^2 = 2, c_2 = 10, q = p_g = 0.$$

Proof. Let s be a fixed point of an order 4 automorphism σ acting on S . As the involution σ^2 has only isolated fix-points, the eigenvalues of σ acting on $T_{S,s}$ cannot be $(i, -1)$ or $(-i, -1)$ (see Lemma 3.5). Let a, b, c be the number of fixed points such that the eigenvalues of σ are (i, i) , $(-i, -i)$ and $(i, -i)$ respectively. The Lefschetz holomorphic fixed point formula implies

$$-\frac{a}{2i} + \frac{b}{2i} + \frac{c}{2} = 1 \text{ and } a + b + c = 4 \text{ or } 2,$$

thus there are two cases :

- 1) $a = b = 1$ and $c = 2$. The singularities of S/G are $2A_{4,1} + 2A_{4,3}$.
- 2) $a = b = 0$ and $c = 2$. In this case, the singularities of S/G are $A_1 + 2A_{4,3}$ because σ^2 has 4 fixed points.

In the first case, a $A_{4,1}$ singularity is resolved by a (-4) -curve C_k . A $A_{4,3}$ singularity is resolved by a chain of three (-2) -curve and we have

$$K_Z = \pi^* K_{S/\sigma} - \sum_{k=1}^{k=2} \frac{1}{2} C_k,$$

thus $K_Z^2 = \frac{8}{4} - 2 = 0$. Additionally,

$$e(S/\sigma) = \frac{1}{4}(4 + (4 - 1)4) = 4,$$

thus $c_2(Z) = 4 + 8 = 12$. The invariants in the second case are computed in a similar way. \square

Proposition 5.7. *Let be $G = \mathbb{Z}/3\mathbb{Z}$. The singularities of quotient surface X are $2A_{3,1} + 2A_{3,2}$. The surface Z has general type and:*

$$K_Z^2 = 2, c_2 = 10, q = p_g = 0.$$

Note that we do not know if Z is minimal in this case.

Proof. We use the notations of Zhang's formula (Proposition 3.4). In this case this formula gives $r_1 + r_2 = 4$. A $A_{3,1}$ singularity is resolved by a (-3) -curve and we have

$$K_Z^2 = \frac{8}{3} - \frac{r_1}{3}.$$

Therefore $r_1 = 2$ and $r_2 = 2$. The singularities of $X = S/\sigma$ are $2A_{3,1} + 2A_{3,2}$. Moreover, as $q = p_g = 0$, we have $c_2 = 10$. Z is of general type by Lemma 5.1. \square

Proposition 5.8. *There is no quaternionic fake quadric with $G = (\mathbb{Z}/3\mathbb{Z})^2 \subset \text{Aut} X$.*

Proof. Let σ_1, σ_2 be the two generating elements of G . Let p be a fixed point of σ_1 . Then as σ_1 and σ_2 commute, $\sigma_2 p, \sigma_2^2 p$ are also fixed points of σ_1 and there are one or four points fixed by the whole group. Let p be such a fixed point. Then there must be a faithful action of G on $T_{S,p}$. But such a faithful action has elements which have eigenvalue 1. This is impossible as an automorphism has only isolated fixed points. \square

Proposition 5.9. *If $G = \mathbb{Z}/6\mathbb{Z}$, then S/G has singularities $2A_{6,1} + 2A_{6,5}$. The minimal resolution Z has invariants:*

$$K_Z^2 = -4, c_2 = 16, q = p_g = 0,$$

Proof. Let s be a fixed point of an order 6 automorphism σ . Let α be a primitive third root of unity. By Lemma 3.5, the action of σ on $T_{S,s}$ has eigenvalues $(-\alpha, (-\alpha)^a)$ or $(-\alpha^2, (-\alpha^2)^a)$ with $a = 1$ or 5 . Let r_1, r_2 and r_3 be respectively the number of fixed points of σ with eigenvalues $(-\alpha, -\alpha)$, $(-\alpha^2, -\alpha^2)$ and $(-\alpha, -\alpha^5)$. Lefschetz fixed point formula (Proposition 3.3) implies the relation

$$\frac{r_1}{(1+\alpha)^2} + \frac{r_2}{(1+\alpha^2)^2} + r_3 = 1,$$

therefore $r_1 = r_2$ and $-r_1 + r_3 = 1$. By Corollary 3.2, σ has 2 or 4 fixed points. The only possibility for (r_1, r_3) is therefore $(1, 2)$. The singularities are $2A_{6,1} + 2A_{6,5}$ and the minimal resolution Z of S/σ has $K_Z^2 = \frac{8}{6} - 2 \cdot \frac{8}{3} = -4$. Moreover $e(Z) = \frac{1}{6}(4 + 5 \cdot 4) + 2 + 2 \cdot 5 = 16$. \square

Proposition 5.10. *If G is the dihedral group \mathbb{D}_3 , the quotient surface S/\mathbb{D}_3 has singularities $4A_1 + A_{3,2} + A_{3,1}$. The minimal resolution Z is of general type and has the following invariants:*

$$K_Z^2 = 1, c_2 = 11, q = p_g = 0.$$

Proof. The group \mathbb{D}_3 is generated by σ and τ such that $\sigma^2 = \tau^3 = (\sigma\tau)^2 = 1$. Every faithful 2 dimensional representation of \mathbb{D}_3 contains reflections, thus by Lemma 3.5, $G = \mathbb{D}_3$ has no fixed point on S .

Let p_1, \dots, p_4 be the fixed points of σ . The fixed points of the involutions $\sigma\tau$ and $\sigma\tau^2$ are respectively $\tau^2 p_i$ and τp_i , $i = 1 \dots 4$. The fixed points of τ are $q_1, q_2 = \sigma q_1$ and $q_3, q_4 = \sigma q_3$. By Proposition 5.7, the images of the q_i in S/G are $A_{3,1} + A_{3,2}$ singularities. Let C be the (-3) -curve on Z over the $A_{3,1}$ -singularity. The images of the $\tau^k p_i$ are 4 A_1 -singularities. We have

$$K_Z = \pi^* K_{S/G} - \frac{1}{3}C,$$

thus $K_Z^2 = \frac{8}{6} - \frac{1}{3} = 1$. By Lemma 3.6, we have

$$e(S/G) = \frac{1}{6}(4 + 2 \cdot 4 + 1 \cdot 12) = 4,$$

and therefore $c_2(Z) = 4 + 1 + 2 + 4 = 11$. \square

Proposition 5.11. *Let be $G = \mathbb{Z}/5\mathbb{Z}$. The singularities of S/G are $4A_{5,2}$, or $A_{5,1} + 2A_{5,2} + A_{5,4}$ or $2A_{5,1} + 2A_{5,4}$. The invariants of the surface Z are respectively:*

$$\begin{aligned} K_Z^2 &= 0, c_2 = 12, \\ K_Z^2 &= -1, c_2 = 13, \\ K_Z^2 &= -2, c_2 = 14, \end{aligned}$$

and in any case $q = p_g = 0$.

Remark 5.12. 1) In Proposition 5.13 below, we give an example of a surface such that the quotient by an order 5 automorphism has $2A_{5,1} + 2A_{5,4}$ singularities.

2) For the same reasons as for $(\mathbb{Z}/3\mathbb{Z})^2$ (see Proposition 5.8), there is no fake quadric X with $(\mathbb{Z}/5\mathbb{Z})^2 \subset \text{Aut} X$.

Proof. Using the notations of Proposition 3.4), the number of fixed points $r_1 + r_2 + r_3 + r_4$ equals 4. As $e(S/\sigma) = \frac{1}{5}(4 + (5-1) \cdot 4) = 4$. Zhang's formula yields

$$(a_1, \dots, a_4) = (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$$

with

$$\sum 4a_i r_i = r_2 + r_3 + 2r_4 = 4.$$

Thus $r_1 = r_4$. Therefore the possibilities for (r_1, r_2, r_3, r_4) are $(0, i, j, 0)$ with $i + j = 4$, or $(1, i, j, 1)$ with $i + j = 2$ or $(2, 0, 0, 2)$. The singularities on the quotient are respectively:

$$\begin{aligned} &4A_{5,2}, \\ &A_{5,1} + 2A_{5,2} + A_{5,4}, \\ &2A_{5,1} + 2A_4. \end{aligned}$$

A singularity $A_{5,i}$ ($i = 1, \dots, 4$) contributes (respectively)

$$-\frac{9}{5}, -\frac{2}{5}, -\frac{2}{5}, 0$$

to K_Z^2 . Thus the self-intersection number is

$$K_Z^2 = \frac{1}{5}(8 - 9r_1 - 2(r_2 + r_3)),$$

and according to the possible tuples (r_1, \dots, r_4) as above: $K_Z^2 = 0$, $K_Z^2 = -1$ and $K_Z^2 = -2$. As $e(S/G) = 4$, we get $c_2 = 12, 13$ or 14 according to the three possible singular loci.

Let us justify our computation of K_Z^2 . A $A_{5,1}$ -singularity is resolved by a (-5) -curve C_5 , thus we have to add $-\frac{3}{5}C_5$ to the canonical divisor. This contributes $(-\frac{3}{5}C_5)^2 = -\frac{9}{5}$ to K_Z^2 . On the other hand, a $A_{5,2}$ -singularity is resolved by a chain of two curves C_2, C_3 with $C_k^2 = -k$. We have to add $-\frac{2}{5}C_3 - \frac{1}{5}C_2$ to π^*K_X , and the contribution to K_Z^2 is

$$(\frac{2}{5}C_3 + \frac{1}{5}C_2)^2 = -\frac{2}{5}.$$

Finally, note that $A_{5,3} = A_{5,2}$ and that the $A_{5,4}$ -singularity doesn't contribute to K_Z^2 . \square

Proposition 5.13. *Let S be a fake quadric with $G = \mathbb{Z}/10\mathbb{Z} \subset \text{Aut}(S)$. The singularities of the quotient surface $X = S/G$ are $2A_{10,1} + 2A_{10,9}$. The resolution Z has the invariants:*

$$K^2 = -12, c_2 = 24, q = p_g = 0.$$

Proof. Let σ be an automorphism of order 10 acting on S . It has 2 or 4 fixed points. As the involution σ^5 has 4 fixed points, it is easy to check that σ cannot have 2 fixed points. Therefore,

$$e(S/G) = \frac{1}{10}(4 + (10 - 1) \cdot 4) = 4.$$

Let ξ be a primitive 5^{th} -root of unity and p a fixed point. There exist $a = a(p)$ and $b = b(p)$ integers invertible mod 5 such that the action of σ on $T_{S,p}$ has eigenvalues $(-\xi^a, -\xi^{ba})$. The Lefschetz holomorphic fixed point formula yields

$$1 = \sum_{p \in S^\sigma} \frac{1}{(1 + \xi^a)(1 + \xi^{ab})}.$$

For $b = 1, 2, 3, 4$, the sum $c(b) = \sum_{a=1}^{a=4} \frac{1}{(1 + \xi^a)(1 + \xi^{ab})}$ is equal to $-4, 1, 1, 6$, respectively. Recall again that $A_{10,3} = A_{10,7}$. For $k \in \{1, 3, 9\}$, let r_k be the number of points in S^σ giving a $A_{10,k}$ singularity. By summing the Lefschetz fixed point

$$4 = -4r_1 + r_3 + 6r_9.$$

Taking care of the relation $r_1 + r_3 + r_9 = 4$, we have the following possibilities for (r_1, r_3, r_9) : $(0, 4, 0)$, $(1, 2, 1)$ and $(2, 0, 2)$.

The resolution of a $A_{10,3}$ -singularity is a chain of 3 curves C_2, C'_2, C_4 with intersection numbers $(-2) - (-2) - (-4)$. We have to add $-\frac{1}{5}(C_2 + C'_2 + C_4)$ to $\pi^*K_{S/G}$. Each singularity contributes $(-\frac{1}{5}(C_2 + C'_2 + C_4))^2 = -\frac{6}{5}$ to K_Z^2 . Similarly, the resolution of a $A_{10,1}$ -singularity is (-10) -curve C_{10} . A $A_{10,1}$ -singularity decreases $K_{S/G}^2$ by $(\frac{-8}{10}C_{10})^2 = -\frac{32}{5}$.

When the singularities of S/G are respectively $4A_{10,3}$, $A_{10,1} + 2A_{10,3} + A_{10,9}$ and $2A_{10,1} + 2A_{10,9}$, we have: $K_Z^2 = \frac{8}{10} - 4\frac{6}{5} = -4$, $K_Z^2 = \frac{8}{10} - \frac{32}{5} - 2\frac{6}{5} - 0 = -8$ and $K_Z^2 = \frac{8}{10} - 2\frac{32}{5} = -12$. The Euler number of Z is respectively $4 + 4 \cdot 2 = 12$, $4 + 1 + 2 \cdot 2 + 9 = 18$ and $4 + 2 + 2 \cdot 9 = 24$. Only the last case is possible because 12 has to divide $K_Z^2 + e(Z)$. \square

Proposition 5.14. *Let S be a fake quadric with $\mathbb{D}_5 \subset \text{Aut}(S)$. The quotient surface by \mathbb{D}_5 has $4A_1 + 2A_{5,2}$ or $4A_1 + A_{5,1} + A_{5,4}$ as singularities. In the first case, the resolution Z has invariants*

$$K^2 = 0, c_2 = 12, q = p_g = 0$$

and is a surface of Kodaira dimension ≥ 1 . In the second case Z has the invariants

$$K^2 = -1, c_2 = 13, q = p_g = 0.$$

Note that we have an example of the second type.

Proof. Let be σ and τ such that $\sigma^2 = \tau^5 = (\sigma\tau)^2 = 1$ generating the group \mathbb{D}_5 acting on S . Suppose that s is a fixed point of \mathbb{D}_5 , then \mathbb{D}_5 acts faithfully on $T_{S,s}$. An involution of a 2 dimensional faithful representation of \mathbb{D}_5 has eigenvalues $(-1, 1)$, but this is impossible because involutions have only isolated fix-points. The 4 fixed points of τ are: $q_1, \sigma q_1, q_2, \sigma q_2$ and if p_1, \dots, p_4 are the 4 fixed points of the involution σ , the fixed points of the involution $\sigma\tau^k$ ($k \in \{0, \dots, 4\}$) are $\tau^{-k}p_1, \dots, \tau^{-k}p_4$. The Euler number of S/\mathbb{D}_5 is:

$$e(S/\mathbb{D}_5) = \frac{1}{10}(4 + (5 - 1) \cdot 4 + (2 - 1) \cdot 20) = 4.$$

The singularities are $4A_1 + 2A_{5,2}$ or $4A_1 + A_{5,1} + A_{5,4}$ and the Chern numbers are respectively $c_1^2 = 0, c_2 = 12$ and $c_1^2 = -1, c_2 = 13$. \square

6. RECONSTRUCTION OF A SURFACE KNOWING ITS QUOTIENT.

In [18], Miyaoka gives a bound on the number of disjoint (-2) -curves on a minimal smooth surface Y . This implies in particular that if $c_1^2 = 4, 2$ or 1 and $\chi(\mathcal{O}_Y) = 1$, there are at most 4, 6 and 7 such curves respectively. The surfaces we obtained as quotient of quaternionic fake quadrics reach that bound. For the cases $c_1^2 = 2, 1$ these surfaces seems to be the first known ones with this property.

In [6] Dolgachev, Mendes Lopes, Pardini study rational surfaces with the maximal number of (-2) -curves. For that aim they use and developpe the theory of $(\mathbb{Z}/2\mathbb{Z})^n$ -covers ramified over A_1 singularities. Using their results, we obtain:

Proposition 6.1. *Let Y be a smooth minimal surface of general type with $q = p_g = 0$ and ${}_2\text{Pic}(Y) = 0$.*

a) If $c_1(Y)^2 = 4$, $c_2(Y) = 8$ and Y contains 4 disjoint (-2) -curves C_1, \dots, C_4 , then there exist a double cover of Y ramified over the curves C_i . The minimal model of this covering has invariants $c_1^2 = 2c_2 = 8$ and $q \leq 1$.

b) If $c_1(Y)^2 = 2$, $c_2(Y) = 10$ and Y contains 6 disjoint (-2) -curves C_1, \dots, C_6 , then there exist a bi-double cover of Y ramified over the curves C_i . The minimal model of this covering has invariants $c_1^2 = 2c_2 = 8$.

a) If $c_1(Y)^2 = 1$, $c_2(Y) = 11$ and Y contains 7 disjoint (-2) -curves C_1, \dots, C_7 , then there exist a $(\mathbb{Z}/2\mathbb{Z})^3$ -cover of Y ramified over the curves C_i . The minimal model of this covering has invariants $c_1^2 = 2c_2 = 8$.

Let \mathbb{F}_2 be the field with 2 elements. Let be C_1, \dots, C_k be k (-2) -curves on a smooth surface Y . Let

$$\psi : \mathbb{F}_2^k \rightarrow \text{Pic}(Y) \otimes \mathbb{F}_2$$

be the homomorphism sending $v = (v_1, \dots, v_k)$ to $\sum v_i C_i$. We say that the curve C_j appears in the kernel $\ker \psi$ if there is a vector $v = (v_1, \dots, v_k)$ in $\ker \psi$ such that $v_j = 1$. For v is in $\ker \psi$ we denote by L_v an element of $\text{Pic}(Y)$ such that $2L_v = \sum v_i C_i$ (we sometimes identify elements of \mathbb{F}_2 to 0, 1 in \mathbb{Z}). We have:

Proposition 6.2. ([6], Proposition 2.3). *Suppose that ${}_2\text{Pic}(Y)$ is zero. There exists a unique smooth connected Galois cover $\pi : Z \rightarrow Y$ such that the Galois group of π is $G = \text{Hom}(\ker \psi, \mathbb{G}_m)$, the branch locus of π is the union of the C_i appearing in $\ker \psi$ and the surface \bar{Z} obtained by contracting the (-1) -curves over the (-2) -curves in Y has invariants:*

$$K_{\bar{Z}}^2 = 2^r K_Y^2 c_2(\bar{Z}) = \chi(\mathcal{O}_{\bar{Z}}) = 2^r \chi(\mathcal{O}_Y) - k2^{r-3}, \kappa(\bar{Z}) = \kappa(Y)$$

where $r = \dim V$.

Proof. (Of Proposition 6.1). We thus have to prove that for our surface Y , $\ker \psi$ has the required dimension and that all the curves appears in $\ker \psi$. For $c_1^2(Y) = 4, 2$ and 1 , we have $b_2(Y) = h^{1,1}(Y) = 6, 8$ and 9 respectively. As we supposed that ${}_2\text{Pic}(Y) = 0$, the space $\text{Pic}(Y) \otimes \mathbb{F}_2$ is $h^{1,1}$ dimensional. As $p_g = 0$, it has moreover a non-degenerate intersection pairing and therefore the dimension of a totally isotropic space in $\text{Pic}(Y) \otimes \mathbb{F}_2$ is at most $\left\lceil \frac{h^{1,1}}{2} \right\rceil = 3, 4, 4$ dimensional

respectively. The image of ψ is the totally isotropic space generated by the curves C_i , therefore the dimension r of $\ker \psi$ is at least 1, 2 and 3 respectively.

A smooth double cover of a surface with n nodes can exist only if n is divisible by 4 (see [6]). Therefore the vectors $v = (v_1, \dots, v_k)$ in $\ker \psi$ (of dimension ≤ 7) have weight 4 i.e. the number of indices j such that $v_j = 1$ is 4.

In case a), $\ker \psi$ is one dimensional, generated by $w_1 = (1, 1, 1, 1)$. For b) and c), as every vector in $\ker \psi$ has weight 4, by [4] Lemme 1, we have $k \geq 2^r - 1$ and thus $r \leq 2$ and $r \leq 3$ respectively. Moreover, it is easy to check that in the case b), the space $\ker \psi$ is (up to permutation of the basis vectors) generated by $w_1 = (1, 1, 1, 1, 0, 0)$ and $w_2 = (1, 1, 0, 0, 1, 1)$. In case c) [4] Lemme 1 implies that $\ker \psi$ is (up to permutation) generated by $w_1 = (1, 0, 0, 1, 1, 0, 1)$, $w_2 = (0, 1, 0, 1, 0, 1, 1)$ and $w_3 = (0, 0, 1, 0, 1, 1, 1)$, thus every curve appears in $\ker \psi$.

The surface is minimal because no surface with $c_1^2 = 3c_2 = 9$ has an order 2 automorphism. \square

Let us give a bound on the irregularity:

Lemma 6.3. *Let Y be a surface of general type with $\chi = 1$ and $q = 0$ containing a 2-divisible set of $4(-2)$ -curves. Let $Y' \rightarrow Y$ be the double cover. Then $q(Y') \leq 1$.*

Proof. As $q(Y) = 0$, the involution σ on Y' given by the cover $Y' \rightarrow Y$ acts by multiplication by -1 on $H^0(Y', \Omega_{Y'})$. Therefore, σ acts trivially on $\wedge^2 H^0(Y', \Omega_{Y'})$. As $p_g(Y) = 0$, the map $\wedge^2 H^0(Y', \Omega_{Y'}) \rightarrow H^0(Y', \wedge^2 \Omega_{Y'})$ must be 0. Let $Y' \rightarrow Y''$ be the blow-down map of the $4(-1)$ -curves over the 4 nodal curves of Y . If $q(Y'') \geq 1$, Castelnuovo-De Franchis Theorem implies that there is a fibration onto a curve B of genus $q(Y'')$. By [28], we get that $q(Y'') \leq 2$ and if $q(Y'') = 2$, then Y'' is an étale bundle of genus 2 fibers onto a genus 2 curve B and $K_{Y''}^2 = 8$. In that case, there is a commutative diagram

$$\begin{array}{ccc} Y'' & \rightarrow & X \\ \downarrow & & \downarrow \\ B & \rightarrow & \mathbb{P}^1 \end{array}$$

where the vertical maps are genus 2 fibrations and X is the surface obtained by contracting the $4(-2)$ -curves on Y . This diagram is obtained from $B \rightarrow \mathbb{P}^1$ by taking base change and normalizing. Since $Y'' \rightarrow X$ is unramified in codimension 1, the 6 fibers of $X \rightarrow \mathbb{P}^1$ occurring at the 6 branch points of $B \rightarrow \mathbb{P}^1$ are double. Since X has only 4 singular points, $X \rightarrow \mathbb{P}^1$ has at least two double fibers contained in the smooth locus of X , but a multiple fiber on a genus 2 fibration cannot exist (because of the adjunction formula). Thus $q \leq 1$. \square

Let us now consider a smooth minimal surface of general type Z with $K^2 = 2$, $c_2 = 10$, $q = p_g = 0$ such that there is a birational map onto a surface Y with singularities $2A_{3,1} + 2A_{3,2}$.

Proposition 6.4. *Suppose that ${}_3\text{Pic}(Z) = 0$. There exists a smooth triple cover X of Y ramified precisely over the singularities of Y . The surface X is of general type and has invariants $c_1^2 = 2c_2 = 8$.*

Proof. Let D_1, D_2 be the (-3) -curves over the 2 singularities $A_{3,1}$ and let D_3, \dots, D_6 be the (-2) curves over the singularities $A_{3,2}$, with indices satisfying: $D_3 D_4 =$

$D_5 D_6 = 1$. Let $W \rightarrow Y$ be the blow up at the intersection points of D_3, D_4 and of D_5, D_6 . Let C_1, \dots, C_6 be the strict transforms of the D_i in W . Let

$$\psi : \mathbb{F}_3^6 \rightarrow \text{Pic}(W) \otimes \mathbb{F}_3 = H^2(W, \mathbb{F}_3)$$

be the homomorphism sending $v = (v_1, \dots, v_k)$ to $\sum v_i C_i$. The image of ψ is a totally isotropic subspace in $H^2(W, \mathbb{F}_3)$. As $b_2(W) = 10$, this image is at most 5 dimensional and therefore $\dim \ker \psi \geq 1$. Let $v = (v_1, \dots, v_6) \in \ker \psi$, $v \neq 0$. We choose the representatives of \mathbb{F}_3 in $\{0, 1, 2\}$. There exist a unique invertible sheaf L such that

$$3L = \sum v_i C_i.$$

Let T be the triple cover of W ramified over the r curves C_i such that $v_i \neq 0$. The surface T is smooth outside the curves C_i with $v_i = 2$. Let R be the minimal resolution of T and let $f : R \rightarrow W$ be the composite map. By [25], Propositions 2.2, 4.1 and 4.3, the invariants of R are:

$$K_R =_{\text{num}} f^*(K_W + \frac{2}{3}\Sigma), \quad c_2(R) = 3c_2(W) - 4r, \quad \chi(\mathcal{O}_R) = 3\chi(\mathcal{O}_W) - \frac{r}{3},$$

where Σ is the sum of the r curves C_i such that $v_i \neq 0$. Therefore $r = 3$ or 6 and

$$K_R^2 = 0, \quad c_2(R) = 36 - 4r, \quad \chi(\mathcal{O}_W) = 3 - \frac{r}{3}.$$

As there are at least 3 curves C_i in the branch locus, one of the curves C_3, \dots, C_6 is in that branch locus. Say it is C_3 . Let E be the exceptional curve going through C_3 . As $C_3 E = C_4 E = 1$ and $E \sum v_i C_i$ is divisible by 3, it forces C_4 to be also in the branch locus and thus $r = 6$ (and $\dim \ker \psi = 1$). The inverse image of the 6 (-3) -curves are (-1) -curves. By the formula giving K_R , the inverse image of the two exceptional curves are (-3) -curves meeting two (-1) -curves. We can therefore effectuate 8 blow-downs and we obtain a fake quadric. It has general type because Y has general type, it is minimal because the quotient of a fake plane by an order 3 automorphism with 4 isolated fixed points has $4A_2$ singularities. \square

Remark 6.5. Let Z be a surface of general type with $c_1^2 = 1$, $q = p_g = 0$. Suppose that there is a birational map onto a surface Y with singularities $4A_1 + A_2 + A_{3,1}$ and $2\text{Pic}(Z) = 0$. It is possible to prove that there exist a $\mathbb{Z}/2\mathbb{Z}$ -cover of Z branched over the 4 nodal curves over the $4A_1$ in Y . If this cover has $q = p_g = 0$ too and no 3 torsion in Pic , then it is possible to construct a smooth surface of general type S with $c_1^2 = 2c_2 = 8$ and a \mathbb{D}_3 -cover $S \rightarrow Y$ branched over the singular points of Y .

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